



Figure 1: Function composition

Notes from 10/5

Feedback on how it's going

- Slow down!
- Confusion about type signatures
- It seems like the GitLab forum is really helpful for those who use it; check it out!

Function composition

Functional programming uses several different operators for combining functions in different ways. The simplest is the *compose* operator, whose syntax is just a dot (period), surrounded by spaces. Here is the mathematical notation, followed by the Haskell notation:

$$(f \circ g)(x) = f(g(x))$$

$$(f . g) x == f (g x)$$

The diagram shows how the two functions are composed, with the output of the *right* one leading to the input of the *left*.

Here are some examples with function composition:

```
ghci> twice x = 2*x
ghci> square x = x*x
ghci> (twice . square) 5
50
ghci> (square . twice) 5
100
```

It can be a little surprising that this works right-to-left, but it's the tradition in the mathematical notation. Another functional language, Elm, uses << for this right-to-left composition, and >> for left-to-right. We can of course define these operators in Haskell too:

```
ghci> (>>) = flip (.)
ghci> (square >> twice) 5
50
ghci> (twice >> square) 5
100
```

We used the built-in `flip` function to reverse the order of the arguments to the compose operator `(.)`.

With partial application

Composition can get a little more interesting when the pieces we're gluing together are themselves the result of partial applications. So here's a function with two arguments that we can partially apply:

```
ghci> f x y = 2*x + 3*y
ghci> g = f 5
ghci> g 7
31
ghci> g 9
37
```

And here's a composition:

```
ghci> (f 5 . f 6 . f 7) 8
388
```

To convince ourselves that this result is correct, we can work through a typical derivation, like:

```
(f 5 . f 6 . f 7) 8 == (Peel off the right-most function)
(f 5 . f 6) (f 7 8) == (Substitute definition of f)
(f 5 . f 6) (2*7 + 3*8) == (Arithmetic)
(f 5 . f 6) 38 == (Repeat)
f 5 (f 6 38) ==
f 5 (2*6 + 3*38) ==
f 5 126 ==
2*5 + 3*126 ==
388
```

With monadic functions

In the [assignment 4](#), we used a different kind of composition operator: `(>=>)`. This is for composing functions that produce *monadic* values, for example, Maybe values. So

applied to Maybe, each function in the pipeline has the option of producing Nothing, in which case that is just returned. If it produces Just, then the value is sent to the next function.

There is also a version (`<=<`) that does right-to-left composition, more like the standard (`.`) operator. To use either of these, you need an import declaration. Below are some examples. We'll define a half function that only works on even numbers; and then a square that refuses to multiply if the number is too large!

```
import Control.Monad
```

```
ghci> half x = if even x then Just (x `div` 2) else Nothing
ghci> square x = if x > 100 then Nothing else Just (x*x)
ghci> (half >=> square) 20
Just 100
ghci> (half >=> square) 240    // square returns Nothing
Nothing
ghci> (half >=> square) 9      // half returns Nothing
Nothing
```

Assignment 4 solutions

The [complete A4 solutions](#) have been posted. In class, we spent a little more time on `isEmpty` and `isFull`, which I defined this way:

```
isFull :: BoundedStack a -> Bool
isFull (BoundedStack cap _) = cap <= 0

isEmpty :: BoundedStack a -> Bool
isEmpty (BoundedStack _ []) = True
isEmpty (BoundedStack _ (_:_)) = False
```

Fullness is based only on the capacity; we don't care about what elements are in the stack. On the other side, emptiness is based only on the elements: they are either the empty list, or there's at least one element (but we don't care what it is).

Interestingly, after presenting this in class, I looked at the implementation I wrote previously, and found that I approached them quite differently. Rather than using the `BoundedStack` constructor to pattern-match, I used the field names `capacity` and `elements`, and then composed them! (`null` is a Boolean function on lists that returns `True` if the list is empty.)

```
isFull :: BoundedStack a -> Bool
isFull = (<= 0) . capacity

isEmpty :: BoundedStack a -> Bool
isEmpty = null . elements
```

Monoids

We previously learned about Functor, which has an `fmap`. If Functor describes something that is “mappable”, then Monoid describes something that is “appendable”. To use Monoids, we need an import:

```
import Data.Monoid
```

A type can be a Monoid if it supports two basic operations:

```
ghci> :i Monoid
class Monoid a where
  mempty :: a
  mappend :: a -> a -> a
```

- `mempty` produces a value of the type that represents nothing, such as an empty list or `Nothing`.
- `mappend` takes two values of the type and combines them together to make a new value that somehow includes both.

List monoid

The simplest demonstration of a Monoid is the list type, where `mempty` is the empty list, and `mappend` is the list concatenation operator (`++`).

```
ghci> mempty :: [Int]
[]
ghci> mappend [1..5] [6..9]
[1,2,3,4,5,6,7,8,9]
```

Like any function with two parameters, you can use `mappend` infix or there's an operator (`<>`) that is equivalent:

```
ghci> [1..5] `mappend` [6..9]
[1,2,3,4,5,6,7,8,9]
ghci> [1..5] <> [6..9]
[1,2,3,4,5,6,7,8,9]
ghci> "Hello" <> "World"
"HelloWorld"
```

There is also a third operation in Monoid, derivable from the other two: `mconcat` takes a list of elements and appends all of them together in sequence:

```
ghci> mconcat ["Alice", "Bob", "Charlie"]
"AliceBobCharlie"
```

Tuple monoid

So that's pretty simple for lists (including strings), but what about other types? A tuple is a monoid if its element types are monoids. For example, here is `mappend` (aka `<>`) applied to a tuples of strings:

```
ghci> ("Alice", "Bob") <> ("Jones", "Smith")
("AliceJones", "BobSmith")
ghci> ("Alice", "Bob", "Carol") <> ("Jones", "Smith", "Patel")
("AliceJones", "BobSmith", "CarolPatel")
```

Question: What is `mempty` when instantiated to a pair of strings?

Maybe monoid

The `Maybe` type is also a monoid, if its element type is a monoid. The `mempty` of course is `Nothing`. And then `mappend` essentially joins the `Just` values, ignoring cases where it finds `Nothing`.

```
ghci> mempty :: Maybe [Int]
Nothing
ghci> mappend (Just [1..4]) (Just [2..8])
Just [1,2,3,4,2,3,4,5,6,7,8]
ghci> mappend (Just [1..4]) Nothing
Just [1,2,3,4]
ghci> mappend Nothing (Just [1..4])
Just [1,2,3,4]
ghci> mappend Nothing Nothing
Nothing
ghci> mconcat [Just "Alice", Nothing, Just "Bob", Just "Carol", Nothing]
Just "AliceBobCarol"
```

Defining a tree monoid

Can we define a `Monoid` instance for a tree? Previously we made a tree data type where both the branches and leaves carried values, of different types. That is a little difficult to reconcile with the expectations of `Monoid`, because there was no such thing as a completely *empty* tree.

So let's design a different kind of tree. In this one, `Leaf` is just an empty constructor, with no value attached at all. Values can be stored at branches. Since each of the left and right subtrees can be an empty `Leaf`, our branches essentially can have zero, one, or two children. (And if there's one child, we can distinguish whether it's a left child or a right child.)

```

data Tree a
  = Leaf
  | Branch { value :: a, left, right :: Tree a }
deriving (Show)

```

Here are a couple of sample trees:

```

sample1 :: Tree String
sample1 =
  Branch "A"
    (Branch "K"
      (Branch "M"
        Leaf
        (Branch "Q" Leaf Leaf))
      (Branch "P" Leaf Leaf))
    Leaf

```

```

sample2 :: Tree String
sample2 =
  Branch "B"
    (Branch "S"
      Leaf
      (Branch "C"
        (Branch "D" Leaf Leaf)
        (Branch "F" Leaf Leaf)))
    (Branch "R" Leaf Leaf)

```

It's helpful if we can print these out in some way. Here's a set of functions I wrote to do some crude ASCII representation of the trees.

```

prettyPrint :: Show a => String -> Tree a -> String
prettyPrint indent Leaf = indent ++ "- *\n"
prettyPrint indent (Branch v Leaf Leaf) =
  indent ++ "- " ++ show v ++ "\n"
prettyPrint indent (Branch v l r) =
  indent ++ "- " ++ show v ++ "\n" ++ prettyPrint tab l ++ prettyPrint tab r
  where tab = indent ++ " |"

printTree :: Show a => Tree a -> IO ()
printTree = putStrLn . prettyPrint ""

```

Their output:

```

ghci> printTree sample1
- "A"
  |- "K"
  |  |- "M"
  |   |- *
  |   |- "Q"
  |   |- "P"
  |   |- *

ghci> printTree sample2
- "B"
  |- "S"
  |  |- *
  |  |- "C"
  |  |- "D"
  |  |- "F"
  |  |- "R"

```

The `*` shows the position of an empty Leaf, but if *both* children are empty leaves we just omit them. The lines and dashes make it pretty easy to see the parent-child relationships in the trees, but here “left” is interpreted as top and “right” as bottom.

What would it mean to append two trees? One way to define it is to require that the value types are also monoids, and then we merge together corresponding branches in the tree structure. In these examples, the roots A and B would merge into AB. Their left children would merge into KS. The first tree’s root doesn’t have a right child, so the result would use the right child of the second tree. Here’s the definition of a Monoid instance:

```

instance Monoid a => Monoid (Tree a) where
  mempty = Leaf
  mappend Leaf Leaf = Leaf
  mappend Leaf br = br
  mappend br Leaf = br
  mappend (Branch v1 l1 r1) (Branch v2 l2 r2) =
    Branch (mappend v1 v2)
           (mappend l1 l2)
           (mappend r1 r2)

```

In the final case – appending two branches – the `(mappend v1 v2)` looks like a recursive call, but it really isn’t. That’s because it’s being used at a different type. So if the tree contains string values, it is appending the strings. When `mappend` is applied to the left and right subtrees, that definitely is recursive.

Here is the result on our two trees, first in one direction:

```
ghci> printTree (sample1 <> sample2)
- "AB"
  |- "KS"
  |  |- "M"
  |   |- *
  |   |- "Q"
  |   |- "PC"
  |   |- "D"
  |   |- "F"
  |- "R"
```

and then the other:

```
ghci> printTree (sample2 <> sample1)
- "BA"
  |- "SK"
  |  |- "M"
  |   |- *
  |   |- "Q"
  |   |- "CP"
  |   |- "D"
  |   |- "F"
  |- "R"
```

Or you can join a tree with itself:

```
ghci> printTree (sample1 <> sample1)
- "AA"
  |- "KK"
  |  |- "MM"
  |   |- *
  |   |- "QQ"
  |   |- "PP"
  |- *
```

Main program

I'm not using this for anything, but if you compile this with `runghc`, it requires a `main`, so here it is.

```
main = return ()
```